Stable multipulse states in a nonlinear dispersive cavity with parametric gain

Stefano Longhi
Centro di Elettronica Quantistica e Strumentazione Elettronica del Consiglio Nazionale delle Ricerche, Dipartimento di Fisica del Politecnico, Piazza L. da Vinci 32, 20133 Milano, Italy
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Existence of stable trains of weakly overlapping solitary waves with phase alternation is predicted in a nonlinear dispersive ring cavity with parametric amplification. It is shown that the breakup of the phase invariance of the wave equation induced by the parametric interaction is effective in damping any internal oscillations of the soliton lattice due to soliton interactions.

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The effects of a parametric interaction on a soliton propagation in nonlinear dispersive media have recently attracted an increasing interest in nonlinear optics both from a fundamental as well from an applicative point of view [1–5]. The use of phase-sensitive amplifiers has been proposed as an effective method for suppressing dispersive soliton radiation [1], Gordon-Haus jitter [2,3], and the Raman self-frequency shift [4] in optical soliton transmission systems. Parametric amplification also reduces the soliton-soliton interaction [3]. When a parametric amplifier is included in a nonlinear dispersive cavity, generation or storage of stable solitary pulses is possible [3,5]. In this case, the cavity field dynamics is governed by a parametric Ginzburg-Landau equation (PGLE), whose dimensionless form is

$$\partial_T u = (-\lambda + i\vartheta)u + \mu u^* + i\partial_t^2 u + i|u|^2 u, \qquad (1)$$

where t is the fast-time variable, T is the slow-time variable describing field evolution at successive round trips, $\lambda > 0$ is the dissipation factor, ϑ is a detuning parameter, and $\mu > 0$ is the parametric gain. Equation (1) represents a general model of pattern forming systems in many physical fields [6]; in particular, it was derived in hydrodynamics as a one-dimensional model to study the onset of parametrically excited waves in the fluid systems [7]. The possibility of generating stable pulse states within the dynamic model described by Eq. (1) is closely related to the existence of a subcritical bifurcation when $\vartheta < 0$ [8]. An exact expression of the solitary waves can be derived as a solution of the PGLE [7,8]:

$$u(t) = \pm\sqrt{2}\beta \mathrm{sech}[\beta(t-\xi)]\exp[i\varphi + i\psi(t-\xi)], \qquad (2)$$

where $\cos(2\varphi) = \lambda/\mu$ with $0 < \varphi < \pi/4$, $\beta^2 = \mu \sin(2\varphi) - \vartheta$, $\psi = 0$, and ξ is an arbitrary constant parameter that reflects the translational invariance of the equation. The double sign in Eq. (2) defines what we will call the ''charge'' (positive or negative) of the soliton. Note that the presence of a phase-sensitive term in the PGLE breaks the phase invariance which is typical of the nonlinear Schrödinger equation, preventing the emergence of a phase-variable term in Eq. (2), as indicated by the condition $\psi = 0$. Stability of the solitary wave requires $\vartheta < 0$ and $\lambda < \mu < \sqrt{\lambda^2 + \vartheta^2}$. In this case, the trivial zero solution is linearly stable, but it can be triggered into a nontrivial state by a finite disturbance or, equivalently, by moving back along the hysteretic loop. Numerical simulations of the PGLE have shown that the modulational unstable stationary state, corresponding to the upper branch of the bifurcating solution, evolves toward irregular complicated temporal patterns [9]. Decreasing the parametric gain, almost periodic stationary states in the form of weakly overlapping solitary waves have been observed [9]. The existence of stationary periodic solutions of the PGLE, of which the solitary wave given by Eq. (2) is a limiting case, was already pointed out by Miles in the context of Faraday waves [7], but their stability was not investigated. The fact that periodic multipulse states are stable attractors of the PGLE is a nontrivial result and a comprehensive physical explanation thereof is needed. In fact, it is well known that the multipulse states of the usual nonlinear Schrödinger equation (such as those arising from the modulational unstable homogeneous state) undergo a periodic wave pattern behavior, which is closely related to the conservative nature of the equation [10]. In terms of a quasiparticle perturbational approach [11], this behavior may be interpreted as due to undamped, internal oscillations of the pulse lattice. When cavity losses are compensated for by use of a linear gain, the achievement of the periodic multipulse states in the cavity requires the use of the external perturbing terms, such as filtering and nonlinear gain or modulation [12].

In this report we show that the stationary periodic solutions in the form of trains of solitary waves with a phase alternation are stable states of the PGLE. The analysis is based on a quasiparticle perturbational approach [11], extensively used to study soliton interactions in the various soliton systems [11–13], and is supported by numerical simulations of the PGLE. Let us preliminarily note that Eq. (1) has a family of stationary periodic solutions given by [7]

$$u(t) = \sqrt{2k}\beta cn(\beta t;k)\exp(i\varphi), \qquad (3)$$

where $cn(\xi;k)$ is an elliptic cosine function of modulus k (the family parameter), $\cos(2\varphi) = \lambda/\mu$ and $\beta^2 = -[\mu \sin(2\varphi) - \vartheta]/(1-2k)$. We assume $\vartheta < 0$, $\sin(2\varphi) > 0$ and $\lambda < \mu < \sqrt{\lambda^2 + \vartheta^2}$, so that the family parameter k may vary in the range $1/2 < k < 1$. Equation (3) represents a periodic function of t with period

$$\tau=\left[\frac{2k-1}{-\vartheta+\mu\,\sin(2\varphi)}\right]^{1/2}\int_0^{2\pi}d\omega[1-k\,\sin^2(\omega)]^{-1}. \quad (4)$$

In the limit $k\to 1^-$, the period τ diverges toward infinity and the solitary wave (2) is recovered. For $k\approx 1$ Eq. (3) may be considered as composed by equally spaced solitary waves with a charge alternation, since the elliptic cosine function changes sign once in a period. On the other hand, when $k\to 1/2$, the period τ goes to zero, the peaks amplitude β diverges, and Eq. (3) describes a wave collapse. Stability analysis of the periodic states (3) is a nontrivial matter and, in general, it is difficult to deal with analytically. However for $k\approx 1$, the periodic state (3) may be approximated as a superposition of weakly overlapping solitary waves, and a quasiparticle perturbational approach may be followed. In general, let us make the ansatz

$$u(t,T)=\sum_n \sqrt{2}\beta\delta_n \mathrm{sech}[\beta(t-\xi_n)]\exp[i\varphi+i\psi_n(t-\xi_n)], \quad (5)$$

where $\delta_n=\pm 1$ is the soliton charge, ξ_n defines the soliton position ($\xi_{n+1}>\xi_n$), and the condition $\beta(\xi_{n+1}-\xi_n)\gg 1$ is assumed to keep valid the weak overlap limit. The effects of the soliton interactions on the internal oscillations of the chain can be taken into consideration by allowing the soliton parameters ψ_n and $d\xi_n/dT$ (which are zero for the single soliton) to become small functions of the slow-time T. It should be noted that, because the solitons are strongly driven by the parametric gain, phases and amplitudes of the solitons are rapidly attracted toward a stationary state that, at zeroth order, corresponds to the single solitary wave. This allows us to consider stationary conditions in Eq. (5) as far as soliton amplitudes and phases are concerned. The coupled equations for the soliton parameters ψ_n and ξ_n may be derived by use of the single-soliton perturbation theory [11]. Because soliton interactions decrease exponentially with soliton separation, only the nearest-neighbor effects are considered. Neglecting higher-order terms, we obtain

$$d\xi_n/dT=\psi_n/m, \quad (6a)$$

$$d\psi_n/dT=-2\mu\,\cos(2\varphi)\psi_n+8\beta^3\{\delta_n\delta_{n+1}\exp[-\beta(\xi_{n+1}-\xi_n)]-\delta_n\delta_{n-1}\exp[-\beta(\xi_n-\xi_{n-1})]\} \quad (6b)$$

where $m=[2-\pi^2\mu\sin(2\varphi)/3\beta^2]^{-1}$. Equations (6) may be regarded as the canonical equations of motion for the interacting charged particles of mass m in the potential

$$U(\xi_1,\xi_2,\ldots)=-\sum_l 8\beta^2\delta_{l+1}\delta_l\exp[-\beta(\xi_{l+1}-\xi_l)] \quad (7)$$

under the action of a viscous force of strength $2\mu\,\cos(2\varphi)=2\lambda$. From Eq. (7) it follows that, as already observed in Ref. [3], solitons with same (opposite) charges attract (repel). The dynamical system (6) has two families of stationary solutions, both corresponding to periodic states with $\xi_{n+1}-\xi_n=\tau$, τ being the common distance between the consecutive solitons. The first one describes a chain of solitons with charge alternation ($\delta_{n+1}=-\delta_n$), and corresponds to the periodic solution (3). The other family is composed by solitons with the same charge ($\delta_{n+1}=\delta_n$). It is expected that only the former configuration may be stable. The period τ of the chain is arbitrary, and only the condition $\beta\tau\gg 1$ must be satisfied in order for the perturbation analysis to be valid. It should also be noted that these stationary states strictly require an infinite number of solitons; however, they can equivalently describe a finite multipulse state circulating in a closed loop, i.e., in an optical cavity. In this case, the soliton spacing is not arbitrary, but may assume a discrete set of values to satisfy the periodic boundary conditions. The stability of the periodic states within the mechanical model of Eqs. (6) can be investigated by the standard linear stability analysis. Linearization of Eqs. (6) around the stationary states yields

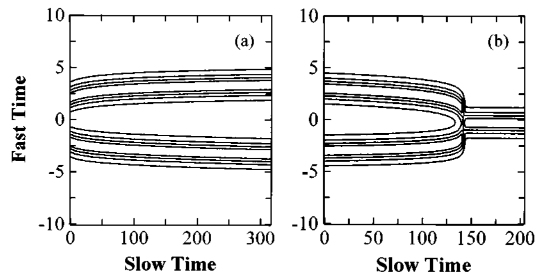

FIG. 1. Contour plot of the two-soliton propagation for (a) solitons with an opposite charge, and (b) for solitons with the same charge. The parameters values are $\lambda=1$, $\mu=1.2$, and $\vartheta=-2$. The initial soliton separation is 4 in (a) and 6 in (b).

$$d\xi_n/dT=\psi_n/m, \quad (8a)$$

$$d\psi_n/dT=-2\lambda\psi_n+mr(\xi_{n+1}+\xi_{n-1}-2\xi_n), \quad (8b)$$

where $r=\mp 8\beta^4\exp(-\beta\tau)/m$ (the upper sign is for solitons with same charge). The secular equation, which determines the eigenvalues σ associated with Eqs. (8), may be expressed in the form $\det[a_{ij}]=0$, where the matrix coefficients are given by $a_{ij}=\sigma^2+2\lambda\sigma+2r$ for $i=j$, $a_{ij}=-r$ for $i=j\pm 1$ and $a_{ij}=0$, otherwise. Stability is ensured provided that real $(\sigma)<0$. For a chain of N solitons and assuming absorbing boundaries, we obtain $\sigma^2+2\lambda\sigma=-4r\sin^2[l\pi/2(N+1)]$, where $l=1,2,\ldots,N$. The condition real$(\sigma)<0$ is satisfied for $r>0$ [14]; therefore the soliton chain with a charge alternation is the only stable state. It is remarkable to note that the viscous force induced by the parametric excitation plays a fundamental role in damping any internal oscillations of the chain, which would be allowed if the dynamical system (6) were conservative. To get further insights into this point, note that, due to the strong damping induced by the viscous force, the momenta ψ_n may be adiabatically eliminated in Eqs. (8). In this way, small motions of the solitons around the stationary state are described by a discrete diffusion equation, which is stable provided that $r>0$. Predictions of the perturbation analysis are well confirmed by the direct numerical simulations of Eq. (1). In Fig. 1, a typical evolution of a soliton pair is reported. It is remarkable that, in Fig. 1(b), the two attracting solitons do not collide elastically, but collapse into a single soliton with the same charge as that of the colliding pulses. During the collapse, half of the field energy is transferred into dispersive waves, which are rapidly

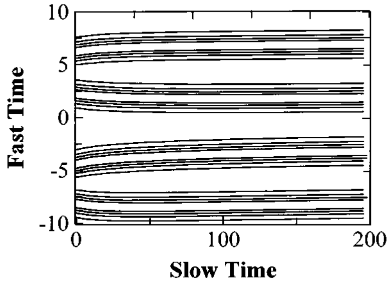

FIG. 2. Contour plot of the propagation of the four-soliton pulses with a charge alternation. The parameters values are $\lambda=1$, $\mu=1.5$, and $\vartheta=-2$.

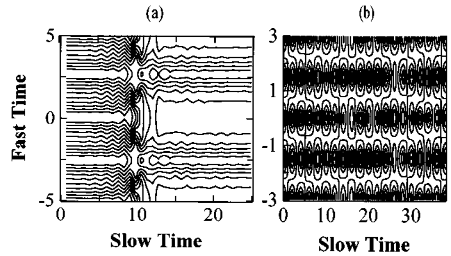

FIG. 3. Propagation of the periodic stationary state given by Eq. (3) for (a) $k=0.9$, and (b) $k=0.7$, with the other parameter values as in Fig. 2. In (a), the instability leads to a breakup of the periodic state into two pulses with the same charge, whereas in (b) it manifests as an irregular oscillation of the lattice.

attenuated in the propagation. This phenomenon is quite general, and the multipulse states formed by solitons with the same charges evolve, after successive collapses, toward a single stationary soliton. On the contrary, we observed that multisoliton states with a charge alternation evolve toward a stationary periodic state, as predicted by the previous analysis. This is shown in Fig. 2 for the case of four differently spaced solitons. In order to simulate periodic transit of the pulses in the cavity, Eq. (1) was integrated assuming periodic boundary conditions. Note that the final soliton spacing is determined solely by the initial solitons number and by the extension of the integration window, which fixes the cavity transit time.

Finally, it is natural to wonder whether Eq. (3) still represents a stable solution of the PGLE beyond the limit of applicability of the perturbation analysis, i.e., beyond the limit $k\approx1$. Numerical integration of Eq. (1) indicates that the stability of the periodic state is lost when the family parameter k is taken away from one. As an example, in Fig. 3(a) the evolution of the field intensity for $k=0.9$ is reported, showing the breakup of the original periodic state into two pulses. Decreasing further the parameter k, we observed an instability which does not lead to the destruction of the periodic state, but results in irregular oscillations of the pulse amplitudes, as shown in Fig. 3(b). It should be noted that the lose of stability when the soliton spacing becomes narrow is probably a very general feature as it was observed in other soliton systems [12,15], and is not strictly related to the PGLE model.

In conclusion, it has been shown both numerically and analytically that periodic chains of parametrically excited solitary pulses with charge alternation may stably propagate in a nonlinear dispersive cavity. Suppression of the internal oscillations of the chain is efficiently achieved by the parametric amplification itself and an introduction of external perturbing terms, such as filtering and nonlinear gain, is not needed. These results may be of interest in nonlinear optics for ultrashort pulse generation and for soliton storage at high frequency.

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